V.Ostrik

TENSOR IDEALS IN THE CATEGORY OF TILTING MODULES.

1. Introduction.

Let $\mathfrak{g}$ be a complex finite dimensional simple Lie algebra with the root datum $(Y, X, \dots)$, see [L2]. Let W_f denote the Weyl group, R denote the root system, R_+ denote the set of positive roots. Let X_+ denote the set of dominant integral weights. Let h denote the Coxeter number of $\mathfrak{g}$.

Let us fix $l \in \mathbb{N}$, $l > h$. We assume that l is odd (and not divisible by 3, if $\mathfrak{g}$ is of type G_2). Let W denote the corresponding affine Weyl group.

Let $\rho \in X$ denote the halfsum of positive roots. We will denote by dot (for example $w \cdot \lambda$) the action of W (and $W_f \subset W$) centered in $(-\rho)$.

Let q be a primitive $l-$th root of unity and let U_q be the quantum group with divided powers as defined in [L2]. Let $\mathcal{C}$ denote the category of finite dimensional U_q-modules of type $\mathbf{1}$ (see e.g. [APW]).

In [A] H.Andersen has studied a tensor subcategory $\mathcal{Q} \subset \mathcal{C}$ formed by *tilting* modules. He has introduced a tensor ideal $\mathcal{K} \subset \mathcal{Q}$ formed by negligible tilting modules. The quotient tensor category $\mathcal{Q}/\mathcal{K}$ is semisimple. For certain values of l it is tensor-equivalent to a category of integrable modules over affine Lie algebra $\hat{\mathfrak{g}}$ equipped with a *fusion* tensor structure (see e.g. [F]).

Let us recall the definition of $\mathcal{K}$. Indecomposable tilting modules are numbered by their highest weights $\lambda \in X_+$; we will denote them by $Q(\lambda)$. The set of dominant weights X_+ is covered by the closed *alcoves* numbered by $W^f \subset W$ — the set of shortest elements in the right cosets W/W_f. For $w \in W^f$ the corresponding closed alcove will be denoted by $\overline{C}_w$. For example, the alcove $\overline{C}_e = \overline{C}$ containing the zero weight is given by

$$\overline{C} = \{\lambda \in X | 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq l \, \text{for all} \, \alpha \in R_+\}.$$

Now $\mathcal{K}$ is formed by the direct sums of tiltings $Q(\lambda)$, where λ is dominant and $\lambda \in \bigcup_{w \neq e} \overline{C}_w$.

Date: November 1996.

This material is based upon work supported by the U.S. Civilian Research and Development Foundation under Award No. RM1-265.

In this note we propose the following generalization of H.Andersen's result. We recall that G.Lusztig and N.Xi have introduced a partition of W^f into *canonical right cells* along with the *right order* $\leq_R$ on the set of cells, see [L1] and [LX]. In particular, $\{e\} \subset W^f$ forms a single right cell, maximal with respect to $\leq_R$. Thus $W^f - \{e\} = \coprod_{A <_R \{e\}} A$ — the union of right cells.

Main Theorem. *Let $A \subset W^f$ be a right cell. The full subcategory $\mathcal{Q}_{\leq A}$ formed by the direct sums of tiltings $Q(\lambda),\ \lambda \in \bigcup_{w \in B \leq_R A} \overline{C}_w$, is a tensor ideal in $\mathcal{Q}$.*

There is a well-known correspondence between the right cells in W and the right ideals in the affine Hecke algebra $\mathcal{H}$ (see [KL1]). Our result is completely parallel to this correspondence, and even the proof is. In fact, the proof is an application of a deep result by W.Soergel who has connected the characters of $Q(\lambda)$ with Kazhdan-Lusztig-type combinatorics of $\mathcal{H}$.

In general, the right cells in W^f are infinite, but some are finite, e.g. $\{e\} \subset W^f$. The first nontrivial example is a "subregular" cell D_1 for $\mathfrak{g}$ of type G_2 (see the pictures and notations in [L1]) consisting of 8 alcoves. Then the subcategory $\mathcal{Q}_{<D_1}$ formed by the direct sums of $Q(\lambda)$ such that $\lambda \in \bigcup_{w \in B <_R D_1} \overline{C}_w$ is a tensor ideal, and we can consider the quotient subcategory $\mathcal{Q}/\mathcal{Q}_{<D_1}$ with finitely many isomorphism classes of indecomposable objects. This subcategory is nonsemisimple, as opposed to Andersen's fusion category $\mathcal{Q}/\mathcal{K}$. For example, when $l = 7$, $\mathcal{Q}/\mathcal{K}$ is equivalent to $\mathbb{C}-$vector spaces, while $\mathcal{Q}/\mathcal{Q}_{<D_1}$ has 24 isomorphism classes of indecomposable objects. Its Grothendieck ring is a 24-dimensional algebra with nontrivial nilpotent radical, as opposed to the classical fusion rings which are always semisimple. To our knowledge, this is a first example of a nonsemisimple tensor category without fiber functor with finitely many indecomposable objects.

As we already mentioned, for certain values of l, $\mathcal{Q}/\mathcal{K}$ is tensor equivalent to a category of integrable $\hat{\mathfrak{g}}-$modules of positive central charge. It is a subcategory of a larger category $\mathcal{O}$ of all $\mathfrak{g}-$integrable $\hat{\mathfrak{g}}-$modules of positive central charge, but the Kazhdan-Lusztig construction of fusion tensor structure in this larger category encounters serious problems (see [KL2]). Still we believe that the quotient categories like $\mathcal{Q}/\mathcal{Q}_{<D_1}$ are closely related to the would-be fusion structure on $\mathcal{O}$.

The idea of this note is essentially due to J.Humphreys : it was he who suggested the important role played by the right cells in the study of tilting modules [H]. I learnt of his ideas from M.Finkelberg. I am grateful to Catharina Stroppel for her beautiful patterns of tilting characters for G_2 which provided a further insight into the connection between right cells and tilting modules. Thanks are also due to

D.Timashov who acquainted me with LIE package; it was very useful for me at the first stage of my work. I am indebted to H.H.Andersen and J.Humphreys for the valuable suggestions which improved the exposition. Finally, I would like to thank the referee for extremely useful comments which simplified the original proof drastically.

2. Preliminaries.

2.1. For any $\lambda \in \overline{C}$ let $\mathcal{C}(\lambda)$ denote a full subcategory of $\mathcal{C}$ consisting of modules whose composition factors have highest weights in $W\cdot\lambda$. The category $\mathcal{C}$ is a direct sum of the subcategories $\mathcal{C}(\lambda)$ (linkage principle; see e.g. [APW], §8)

$$\mathcal{C} = \bigoplus_{\lambda\in\overline{C}} \mathcal{C}(\lambda).$$

For any $\lambda \in X_+$ one defines Weyl module $V(\lambda)$ and module $H^0(\lambda)$ (see [A] §1). Then the irreducible module $L(\lambda)$ is the socle of $H^0(\lambda)$ as well as the head of $V(\lambda)$.

2.2. Let $\mathbb{Z}[X]$ be the group algebra of abelian group X. It is generated by elements e^λ, $\lambda \in X$, with relations $e^{\lambda_1}\cdot e^{\lambda_2} = e^{\lambda_1+\lambda_2}$ $\forall \lambda_1, \lambda_2 \in X$. There is a natural action of W_f on $\mathbb{Z}[X]$ given by the formula $we^\lambda = e^{w\lambda}$. Let $\mathcal{A} := \mathbb{Z}[X]^{W_f}$ be the invariants of this action. It is a subalgebra of $\mathbb{Z}[X]$.

Let $ch : K(\mathcal{C}) \to \mathbb{Z}[X]$ be the map associating to a module $M \in \mathcal{C}$ its character $ch(M)$. It is known that its image is $\mathcal{A}$. Moreover the elements $ch([V(\lambda)])$ where λ runs through X_+ form a basis of $\mathcal{A}$. It is known that $ch([V(\lambda)]) = ch([H^0(\lambda)])$ is given by the Weyl character formula (see e.g. [APW] §8):

$$ch([V(\lambda)]) = \frac{\sum_{w\in W_f}(-1)^{l(w)}e^{w\cdot\lambda}}{\sum_{w\in W_f}(-1)^{l(w)}e^{w\cdot 0}}.$$

Now for any $\lambda \in X$ let

$$ch(\lambda) = \frac{\sum_{w\in W_f}(-1)^{l(w)}e^{w\cdot\lambda}}{\sum_{w\in W_f}(-1)^{l(w)}e^{w\cdot 0}}.$$

Lemma. (i)*If stabilizer (in W_f) of λ with respect to dot action is nontrivial then $ch(\lambda) = 0$.*

(ii)*Suppose the stabilizer of λ is trivial and let $w \in W_f$ be such that $w\cdot\lambda \in X_+$. Then $ch(\lambda) = (-1)^{l(w)}ch(w\cdot\lambda)$.*

Proof. Clear. □

2.3. Let $W \to W_f$, $w \mapsto \overline{w}$ be the standard homomorphism with the kernel consisting of translations.

Lemma. *For any $\lambda, \mu \in X$ and $w \in W$ we have $w(\lambda+\mu) = w\lambda + \overline{w}\mu$ and $w \cdot (\lambda+\mu) = w \cdot \lambda + \overline{w}\mu$.*

Proof. The first identity is obviously true for $w \in W_f \subset W$ and for translations. Since W is a semidirect product of W_f and the subgroup of translations we get our result. The second identity is a simple consequence of the first one. □

2.4. **Lemma.** (see e.g. [D] 2.2.3) *Let P be a multiset (set with multiplicities) of weights invariant under W_f action. Then for any $\lambda \in X$ we have*

$$(\sum_{\omega \in P} e^{\omega}) ch(\lambda) = \sum_{\omega \in P} ch(\lambda + \omega)$$

Proof. Straightforward computation. □

2.5. A filtration of U_q-module is called *Weyl filtration* (respectively *good filtration*) if all the associated factors are Weyl modules (respectively modules $H^0(\lambda)$).

2.6. **Definition** (see [A], definition 2.4) *A tilting module is a module $M \in \mathcal{C}$ which has both a Weyl filtration and a good filtration.*

Let $\mathcal{Q} \subset \mathcal{C}$ be a full subcategory formed by all tilting modules. The main properties of this category are collected in the following (see [A] §2)

Theorem. (i) *The category $\mathcal{Q}$ is closed under tensor multiplication.*

(ii) *Any tilting module is a sum of indecomposable tilting modules.*

(iii) *For each $\lambda \in X^+$ there exists an indecomposable tilting module $Q(\lambda)$ with highest weight λ.*

(iv) *The modules $Q(\lambda), \lambda \in X^+$, form a complete set of nonisomorphic indecomposable tilting modules*

(v) *A tilting module is determined up to isomorphism by its character.*

Let $\mathcal{Q}(\lambda)$ be the full subcategory of $\mathcal{Q}$ consisting of modules contained in $\mathcal{C}(\lambda)$. Then obviously

$$\mathcal{Q} = \bigoplus_{\lambda \in \overline{C}} \mathcal{Q}(\lambda).$$

2.7. For any $\lambda, \mu \in \overline{C}$ one defines the translation functor $T_\lambda^\mu : \mathcal{C}(\lambda) \to \mathcal{C}(\mu)$ (see e.g. [APW] §8). The following Proposition is proved as in [J],II,7.13.

2.7.1. **Proposition.** *Suppose $\lambda, \mu \in \overline{C}$ and $w \in W$ is such that $w \cdot \lambda \in X_+$. Then $T_\lambda^\mu V(w \cdot \lambda)$ has a filtration with the associated factors $V(\nu)$ such that $\nu \in X_+$ and $\nu = ww_1 \cdot \mu$ with $w_1 \in Stab(\lambda)$. Each one of the above factors occurs exactly once.*

In particular it follows that translation functors preserve the category $\mathcal{Q}$.

2.7.2. **Corollary.** *For any $w \in W$ such that $w \cdot \lambda \in X_+$ the module $T_\lambda^0 T_0^\lambda V(w \cdot 0)$ has a filtration with associated factors $V(wx \cdot 0)$ with $x \in Stab(\lambda)$.*

Proof. Evident. □

3. Construction of tensor ideals.

3.1. Recall that W denotes the affine Weyl group and W_f denotes the ordinary Weyl group. Let W^f denote the set of minimal length representatives of right cosets. The multiplication defines a bijection $W_f \times W^f \to W$. Let $\mathfrak{L}$ be the sign representation of W_f. We will consider it as right W_f-module. Let us define a right $W-$module $\mathcal{N}^1 := \mathfrak{L} \otimes_{\mathbb{Z}[W_f]} \mathbb{Z}[W]$. As $\mathbb{Z}-$module it is isomorphic to a free abelian group with generators numbered by W^f. Let $N_x^1 = 1 \otimes x$ for any $x \in W^f$. These elements form a $\mathbb{Z}-$basis of $\mathcal{N}^1$. For any $s \in S$ we have $N_x^1 s = N_{xs}^1$ if $xs \in W^f$ and $N_x^1 s = -N_x^1$ otherwise.

3.2. Let $K(\mathcal{C})$ denote the Grothendieck group of the category $\mathcal{C}$. For any $\lambda \in \overline{C}$ define the map $\alpha_\lambda : K(\mathcal{C}) \to \mathcal{N}^1$ by $\alpha_\lambda([V(\mu)]) = 1 \otimes (\sum_{x \in W, x \cdot \lambda = \mu} x)$. In particular α_λ annihilates every object outside of the block $\mathcal{C}(\lambda)$ of $\mathcal{C}$.

3.2.1. Let us identify $K(\mathcal{C})$ with the character ring $\mathcal{A}$.

Lemma. *For any $w \in W$ we have*

$$\alpha_\lambda(ch(w \cdot \lambda)) = |Stab(\lambda)|^{-1} 1 \otimes (\sum_{x \in Stab(\lambda)} wx)$$

Proof. For $w \in W$ such that $w \cdot \lambda \in X_+$ the Lemma is clear from definitions. For other w use 2.2. □

3.2.2. **Lemma.** *For any $\lambda \in \overline{C}$ and $V \in \mathcal{C}(\lambda)$ we have*

$$\alpha_\lambda(V) = \alpha_0(T_\lambda^0 V)$$

Proof. Obvious. □

3.2.3. **Lemma.** *For any $\lambda \in \overline{C}$ and $V \in \mathcal{C}(0)$ we have*

$$\alpha_0(T_\lambda^0 T_0^\lambda V) = \alpha_0(V) \sum_{x \in Stab(\lambda)} x$$

Proof. It is enough to verify the Lemma for $V = V(w \cdot 0)$. Now if $w \cdot \lambda \in X_+$ the result follows from 2.7.2; if $w \cdot \lambda \notin X_+$ then RHS and LHS both vanish. □

3.3. **Proposition.** *For any $\lambda, \mu \in \overline{C}$ and $M \in \mathcal{C}$ there exists $c(M) = c_{\lambda\mu}(M) \in \mathbb{Z}[W]$ such that for all $V \in C(\lambda)$ we have*

$$\alpha_\mu(V \otimes M) = \alpha_\lambda(V)c(M).$$

Proof. (see also [J],II,7.5) It is enough to check the claim on the level of characters; moreover we can suppose that $ch(V) = ch(w \cdot \lambda)$.

Let $P(M)$ be a multiset of weights of module M. It is invariant under W_f-action. We have by 2.4 and 2.3

$$ch(V(w \cdot \lambda) \otimes M) = \sum_{\omega \in P(M)} ch(w \cdot \lambda + \omega) = \sum_{\omega \in P(M)} ch(w \cdot (\lambda + \omega))$$

Now let us define a multiset $W_{\lambda\mu}(M) := \{x \in W | \lambda + \omega = x \cdot \mu; \omega \in P(M)\}$ It is easy to see that $W_{\lambda\mu}(M)$ is invariant under left multiplication by elements of $Stab(\lambda)$ and right multiplication by elements of $Stab(\mu)$. So $W_{\lambda\mu}(M)$ is a union of left and right cosets; let $W_{\lambda\mu}(M)'$ be a set of representatives of right cosets. We claim that we can choose $c_{\lambda\mu}(M) = \sum_{z \in W_{\lambda\mu}(M)'} z$.

Indeed, let $P_{\lambda\mu}(M) := \{\omega \in P(M) | \lambda + \omega \in W \cdot \mu\}$. For any $\omega \in P_{\lambda\mu}(M)$ let $w(\omega)$ be any element of W such that $w(\omega)^{-1} \cdot (\lambda + \omega) = \mu$. It is evident that $\{w(\omega)\}$ is the set of representatives of left cosets in $W_{\lambda\mu}(M)$. We have

$$\alpha_\mu(V \otimes M) = \alpha_\mu(ch(w \cdot \lambda)ch(M)) = \alpha_\mu(\sum_{\omega \in P(M)} ch(w \cdot (\lambda + \omega))) =$$

$$\alpha_\mu(\sum_{\omega \in P_{\lambda\mu}(M)} ch(w \cdot (\lambda + \omega))) = \alpha_\mu(\sum_{\omega \in P_{\lambda\mu}(M)} ch(ww(\omega) \cdot \mu)) =$$

$$\sum_{x \in Stab(\mu)} \sum_{\omega \in P_{\lambda\mu}(M)} 1 \otimes ww(\omega)x = \sum_{t \in W_{\lambda\mu}(M)} 1 \otimes wt =$$

$$\sum_{y \in Stab(\lambda)} \sum_{z \in W_{\lambda\mu}(M)'} 1 \otimes wyz = \alpha_\lambda(ch(w \cdot \lambda)) \sum_{z \in W_{\lambda\mu}(M)'} z$$

The Proposition is proved. □

3.4. **Definition.** *A subcategory $\mathcal{C}' \subset \mathcal{C}$ is called a weak tensor ideal if for any $V \in \mathcal{C}'$ and $M \in \mathcal{C}$ we have $V \otimes M \in \mathcal{C}'$.*

We define weak tensor ideals in any subcategory of $\mathcal{C}$ closed under tensor multiplication in the same way.

Corollary. *If $U \subset \mathcal{N}^1$ is a $\mathbb{Z}[W]-$submodule, then $\mathcal{C}_U := \{V \in \mathcal{C} | \alpha_\lambda(V) \in U \; \forall \lambda \in \overline{C}\}$ is a weak tensor ideal of $\mathcal{C}$ and $\mathcal{Q}_U := \mathcal{Q} \cap \mathcal{C}_U$ is a weak tensor ideal of $\mathcal{Q}$.*

Proof. Clear. □

4. Realization of $K(\mathcal{Q}(0))$ as a module over Hecke algebra.

In this section we follow [S1] .

4.1. Let $l : W \to \mathbb{N}$ be the length function and let $\leq$ be the standard Bruhat order on W. We will write $x < y$ if $x \leq y$ and $x \neq y$. Let $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$ denote the ring of Laurent polynomials over $\mathbb{Z}$ in variable v. Let $\mathcal{H}$ be the Hecke algebra corresponding to (W, S)

$$\mathcal{H} = \bigoplus_{x \in W} \mathcal{L} T_x$$

with multiplication given by the rule: $T_x T_y = T_{xy}$ if $l(xy) = l(x) + l(y)$ and $T_s^2 = v^{-2} T_e + (v^{-2} - 1) T_s$ for all $s \in S$ (see [S1] §2).

Let $H_x = v^{l(x)} T_x$ be a new basis of Hecke algebra. There exists unique involutive automorphism of Hecke algebra $d : \mathcal{H} \to \mathcal{H}, H \mapsto \overline{H}$ such that $\overline{v} = v^{-1}$ and $\overline{H_x} = (H_{x^{-1}})^{-1}$. We will call $H \in \mathcal{H}$ selfdual if $\overline{H} = H$.

The following theorem was proved by Kazhdan and Lusztig in [KL1].

Theorem. *For any $x \in W$ there exists unique selfdual $\underline{H}_x \in \mathcal{H}$ such that $\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v] H_y$.*

The coefficients of $\underline{H}_x$ in the basis $\{H_x\}$ are essentially Kazhdan-Lusztig polynomials.

4.2. Let $\mathcal{H}_f$ be the Hecke algebra corresponding to the group W_f. We have an obvious embedding $\mathcal{H}_f \subset \mathcal{H}$. Let $\mathcal{L}(-v)$ be a free right $\mathcal{L}-$module of rank 1 with the right action of $\mathcal{H}_f$ given by the following rule: for any $s \in S_f$ the element H_s acts as $(-v)$. We define a right $\mathcal{H}-$module $\mathcal{N} := \mathcal{L}(-v) \otimes_{\mathcal{H}_f} \mathcal{H}$. For any $x \in W^f$ let us define $\underline{N}_x := 1 \otimes \underline{H}_x \in \mathcal{N}$. Let $\beta : \mathcal{N} \to \mathcal{N}^1$ denote the specialization map: $v \mapsto 1$. We define $\underline{N}_x^1 := \beta(\underline{N}_x) \in \mathcal{N}^1$.

4.3. The following statement was conjectured in [S1] (Vermutung 7.2) and then proved in [S2].

Theorem. $\alpha(Q(x \cdot 0)) = \underline{N}_x^1$.

4.4. We will say that an $\mathbb{Z}[W]-$submodule of $\mathcal{N}^1$ is a *KL-submodule* if it admits a base consisiting of elements $\underline{N}^1_x$ for some subset of W^f.

5. Right cells in affine Weyl group.

5.1. In [KL1] Kazhdan and Lusztig defined three partitions of any Coxeter group into subsets called right, left and two-sided cells respectively. We refer the reader to *loc. cit.* for the definitions of preorders $\leq_R, \leq_L, \leq_{LR}$ on Coxeter groups. The right (left, two-sided) cells are the classes of equivalence generated by preorder $\leq_R$ (respectively $\leq_R$ and $\leq_{LR}$). Let $w \in W$ and A be a right cell in W. We will write that $w \leq_R A$ if $w \leq_R w'$ for any $w' \in A$ (and similarly for left and two-sided cells).

5.2. There is a correspondence between cells and ideals in the Hecke algebra. Namely, for any right (left or two-sided) cell A the $\mathcal{L}-$submodule $I_{\leq A}$ of $\mathcal{H}$ generated by $\underline{H}_w$, $w \leq_R A$ (and similarly for left and two-sided cells) is a right (respectively left and two-sided) ideal of $\mathcal{H}$ (see [KL1]). Moreover any KL-ideal (i.e. ideal admiting a base consisting of some elements $\underline{H}_w$) is a sum of such ideals.

5.3. Let A be a two-sided cell of W. The main result of [LX] is the following

Theorem. *The intersection $A \cap W^f$ forms a right cell of W.*

5.4. **Definition.** *A weak tensor ideal $\tau \subset \mathcal{Q}$ is called a tensor ideal if for any Q_1, Q_2 such that $Q_1 \oplus Q_2 \in \tau$ we have $Q_1, Q_2 \in \tau$.*

For any two-sided cell A of W we define the full subcategory $\mathcal{Q}_{\leq A}$ of $\mathcal{Q}$ as follows: $\mathcal{Q}_{\leq A}$ is the additive subcategory of $\mathcal{Q}$ and indecomposable objects of $\mathcal{Q}_{\leq A}$ are all the modules $Q(w \cdot \lambda)$ where $\lambda \in \overline{C}$, $w \in W^f$ and $w \leq_R A$.

5.5. **Main Theorem.** *For any two-sided cell A of W the subcategory $\mathcal{Q}_{\leq A}$ is a tensor ideal.*

Proof. For any two-sided cell A we define a $\mathbb{Z}[W]-$submodule $U_{\leq A}$ of $\mathcal{N}^1$ to be $\mathcal{L} \otimes I_{\leq A \cap W^f}$.

We will show that for any $\lambda \in \overline{C}$ $\alpha_\lambda(Q(w \cdot \lambda)) \in U_{\leq A}$ if and only if $\alpha_0(Q(w' \cdot 0) \in U_{\leq A}$ where w' is the longest element of coset $wStab(\lambda)$. We have

$$\alpha_\lambda(Q(w \cdot \lambda)) = |Stab(\lambda)|^{-1} \alpha_0(T^0_\lambda Q(w \cdot \lambda))$$

Note that $T^0_\lambda Q(w \cdot \lambda)$ contains a direct summand $Q(w' \cdot 0)$. So we proved that $\alpha_\lambda(Q(w \cdot \lambda)) \in U_{\leq A}$ implies that $\alpha_0(Q(w' \cdot 0) \in U_{\leq A}$.

Now note that $T_0^\lambda Q(w' \cdot 0)$ contains a direct summand $Q(w' \cdot \lambda) = Q(w \cdot \lambda)$. Further $\alpha_0(T_\lambda^0 T_0^\lambda Q(w' \cdot 0)) = \alpha_0(Q(w' \cdot 0)) \sum_{x \in Stab(\lambda)} x \in U_{\leq A}$ by 3.2.3 and we proved our claim in another direction.

So the proof of theorem is finished. □

5.6. *Remark.* It is easy to see that theorem above establishes bijection between KL-submodules of $\mathcal{N}^1$ and tensor ideals in $\mathcal{Q}$. Further note that all KL-submodules of $\mathcal{N}^1$ are the sums of submodules $U_{\leq A}$. So we describe all tensor ideals in a category of tilting modules.

References

[A] H.H.Andersen *Tensor products of quantized tilting modules.* Communications in Mathematical Physics 149 (1992), pp. 149-159.

[APW] H.Andersen, P.Polo, K.Wen *Representations of quantum algebras.* Invent. Math. 104 (1991), pp.1-59.

[D] S.Donkin *Rational representations of algebraic groups.* Lecture Notes in Math. vol 1140, Berlin Heidelberg, New York: Springer 1985.

[F] M.Finkelberg *An equivalence of fusion categories*, Geometrical and Functional Analysis, vol.6, 2 (1996), pp. 249-267.

[H] J.E. Humphreys, *Comparing modular representations of semisimple groups and their Lie algebras*, in volume dedicated to R.E. Block, Modular Interfaces: Modular Lie Algebras, Quantum Groups, and Lie Superalgebras, ed. V. Chari and I. Penkov, International Press, 1997.

[J] J.C.Jantzen *Representations of Algebraic Groups*, Pure and Applied Mathematics 131, Academic Press, 1987.

[KL1] D.Kazhdan, G.Lusztig *Representations of Coxeter groups and Hecke algebras* Inventiones math. 53 (1979), pp. 165-184.

[KL2] D.Kazhdan, G.Lusztig *Tensor structures arising from affine Lie algebras I-IV*, J.Amer.Math.Soc. 6 (1993), pp. 905-1011 and J.Amer.Math.Soc. 7 (1994), pp. 335-453.

[L1] G.Lusztig *Cells in affine Weyl groups.*, Advanced Studies in Pure Mathematics 6, 1985; Algebraic Groups and Related Topics pp. 255-287.

[L2] G.Lusztig *Introduction to quantum groups.*, Boston, Birkhauser, 1993.

[LX] G.Lusztig, N.Xi *Canonical left cells in affine Weyl groups,* Advances in Mathematics 72 (1988), pp. 284-288.

[S1] W.Soergel *Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln.*, Representation Theory, 1 (1997).

[S2] W.Soergel *Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren.*, Preprint (1996), pp. 1-21.

Independent Moscow University, 11 Bolshoj Vlasjevskij per., Moscow 121002 Russia

E-mail address: `ostrik@nw.math.msu.su`